TRANSMISSION OF RADIATION BY A MEDIUM IN TWO-DIMENSIONAL PROBLEMS

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General formulas for the transmission characteristics of a medium for the simplest two-dimensional systems with an exponential law of absorption are given. These characteristics are the coefficients of irradiation, the coefficients of spatial distribution of the incident flux, and the coefficient of utilization of a point source. The emission characteristic of real surfaces is expressed by a cosine power series.

The transfer of radiant energy in a medium is usually calculated with allowance for the diffusivity of the fluxes incident on the medium. Yet the intrinsic or effective emission even of ordinary surfaces does not conform to a cosine law. For a semitransparent body with a temperature varying with depth, the emission of the surface may deviate very considerably from a cosine law. If the surface is covered by sources (for instance, lamps in the case of drying with infrared rays), the emission characteristic depends on the type of source. Very diverse characteristics are given by fluxes of nuclear radiation and molecular fluxes.

The (effective or intrinsic) emission characteristic of an element of surface can conveniently be expressed by a series converging in the interval $[0, \pi/2]$ [1]:

$$f(\Theta) = \sum_{n=0}^{\infty} a_n \cos^n \Theta,$$

where

$$\sum_{n=0}^{\infty} a_n = 1.$$

In some cases it is sufficient to take the first three terms of the series with odd powers of the cosine [2]. Another characteristic of the emitting element will be the equivalent solid angle Ω [2], which is the solid angle in which an isotropic (in a hemisphere) flux of the same intensity can be propagated at $\Theta = 0$ (normal to the element):

$$\Omega = \int_{2\pi} f(\Theta)\, d\omega = 2\pi \sum_{n=0}^{\infty} \frac{a_n}{n+1}. \tag{1}$$

Henceforth we assume an exponential absorption law and consider only two-dimensional problems, for which the special $Ki_n(x)$ functions, used earlier in calculations of convective heat transfer [3] and radiation dosimetry [4], can be successfully used. Functions $Ki_n(x)$ have been tabulated for n from 1 to 16 in the range $0 \le x \le 3$ [5].

Below we give the essential properties of a $Ki_n(x)$ function:

$$Ki_n(x) = \int_0^{\pi/2} \exp(-x/\cos\alpha) \cos^{n-1}\alpha\, d\alpha,$$

$$dKi_n(x) = -Ki_{n-1}(x)\, dx, \quad Ki_n(x) = \int_x^{\infty} Ki_{n-1}(t)\, dt,$$

or

$$Ki_n(x) = Ki_n(0) - \int Ki_{n-1}(x)\, dx, \tag{2}$$

where $Ki_n(0)$ is a constant of integration.

The recurrence relation

$$(n+1) Ki_{n+2}(x) = n Ki_n(x) + x\,[Ki_{n-1}(x) - Ki_{n+1}(x)] \quad \text{for} \quad n \geqslant 1, \tag{3}$$

$$Ki_n(\infty) = 0, \quad Ki_n(0) = \sqrt{\pi}\,\Gamma(n/2)/2\Gamma[(n+1)/2].$$

For instance,

$$Ki_1(0) = \pi/2, \quad Ki_2(0) = 1, \quad Ki_3(0) = \pi/4, \quad Ki_4(0) = 2/3,$$

$$-\int Ki_n(x) \frac{dx}{x^{n-1}} = \frac{Ki_n(x)}{(n-2)\,x^{n-2}} + \int Ki_{n-1}(x) \frac{dx}{(n-2)\,x^{n-2}} \quad \text{for } n > 2, \tag{4}$$

$$\int Ki_2(x) \frac{dx}{x} = \frac{1}{x}[Ki_1(x) - 2Ki_3(x)], \tag{5}$$

$$\text{as} \quad x \to 0 \quad \int Ki_2(x) \frac{dx}{x} \to 0.8840685 + \ln x. \tag{6}$$

The connection with modified Bessel functions of imaginary argument are

$$Ki_0(x) \equiv K_0(x),$$

$$Ki_1(x) = \int_x^{\infty} K_0(u)\, du,$$

$$Ki_2(x) = x\left[K_1(x) - \int_x^{\infty} K_0(u)\, du\right],$$

$$2Ki_3(x) = (1 + x^2) \int_x^{\infty} K_0(u)\, du + xK_0(x) - x^2 K_1(x).$$

The last and further relations are difficult to obtain from the recurrence relation (3).

The transmission of radiation by a medium is characterized by several quantities.

Coefficient of irradiation. This represents the density of the direct flux at a point dF_1 in the case of unit hemispherical effective flux from surface F_2 (Fig. 1). The direct flux includes particles which have not interacted with the medium:

$$\xi_{12} = \frac{1}{\Omega} \int_{F_2} \exp(-kl) f(\Theta_2) \cos\Theta_1\, dF_2/l^2.$$

By means of the substitutions $\cos\Theta_2 dF_2/l^2 = d\omega = \cos\alpha d\alpha d\beta$, $kl = x/\cos\alpha$, $\cos\Theta_1 = \cos\alpha\cos\beta$ we obtain

$$\xi_{12} = \frac{2}{\Omega}\int_{\beta_1}^{\beta_2}\cos\beta\, d\beta\int_0^{\pi/2}\exp\left(-\frac{x}{\cos\alpha}\right)\frac{f(\Theta_2)}{\cos\Theta_2}\cos^2\alpha\, d\alpha. \quad (7)$$

Mean coefficient of irradiation:

$$\varphi_{12} = \frac{1}{L}\int_L \xi_{12}\, dL,$$

where L is the extent of the two-dimensional surface.

Exact calculation of the mean coefficients of irradiation is possible only in the simplest examples. They should be calculated approximately from several values of the local coefficients of irradiation.

Coefficient of spatial distribution of incident flux. The density of the bulk incident flux at point dF_1 is given by the formula

$$\eta_{i1} = \frac{q_{eff2}}{\Omega}\int_{F_2}\exp(-kl)\frac{f(\Theta_2)\, dF_2}{l^2}.$$

The quantity

$$m = \eta_{i1}/\xi_{12} q_{eff2}$$

shows the spatial distribution of the direct flux incident on the point dF_1. It is of interest in the case of irradiation of a surface with relief. When all the incident rays are normal to the element of area, m = 1.

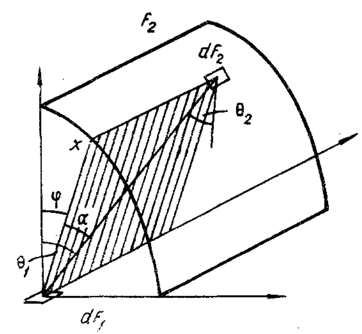

Fig. 1. General diagram showing position of surface element dF_1 and surface F_2.

In the case of a diffuse incident flux, m = 2. If the mean angle of incidence tends to 90°, then $m \to \infty$. We will later give formulas for the product $m_1\xi_{12}$,

$$m_1\xi_{12} = \frac{2}{\Omega}\int_{\beta_1}^{\beta_2} d\beta\int_0^{\pi/2}\exp\left(-\frac{x}{\cos\alpha}\right)\frac{f(\Theta_2)}{\cos\Theta_2}\cos\alpha\, d\alpha. \quad (8)$$

Coefficient of utilization of point axisymmetric emitter. Here we will continue the work of [6], but for two-dimensional problems and with due regard to the absorption of the medium. Let a point source with emission characteristic $f(\Theta_1)$ be placed at point dF_1 (Figs. 1, 2). We require the determination of its coefficient of utilization for two-dimensional surfaces (for a direct flux). It is easy to arrive at the general formula

$$u_1 = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n\int_{\beta_1}^{\beta_2} Ki_{n+2}(x)\cos^n\beta\, d\beta. \quad (9)$$

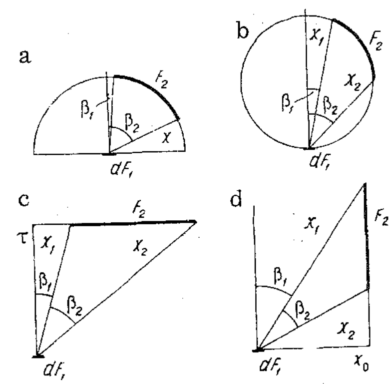

Fig. 2. Cross sections of simple semitransparent bodies.

Since

$$\text{for } x = 0, \quad \beta_1 = -\pi/2, \quad \beta_2 = \pi/2, \quad u_1 = 1,$$

then, in addition to (1),

$$\Omega = 4\sum_{n=0}^{\infty} a_n Ki_{n+2}(0)\int_0^{\pi/2}\cos^n\beta\, d\beta.$$

Figure 2 shows the simplest cases, for which formulas (7)–(9) become very much simpler.

Case A. $\cos\Theta_2 = \cos\alpha$, x is the optical (as regards attenuation of the ray) radius.

$$\xi_{12} = (2/\Omega)(\sin\beta_2 - \sin\beta_1)\sum_{n=0}^{\infty} a_n Ki_{n+2}(x),$$

$$m_1\xi_{12} = (2/\Omega)(\beta_2 - \beta_1)\sum_{n=0}^{\infty} a_n Ki_{n+1}(x),$$

$$u_1 = (2/\Omega)\sum_{n=0}^{\infty} a_n Ki_{n+2}(x)\int_{\beta_1}^{\beta_2}\cos^n\beta\, d\beta.$$

Case B. $\cos\Theta_2 = \cos\alpha\cos\beta$, $x = D\cos\beta$, where D is the optical (as regards attenuation of the ray) diameter.

$$\xi_{12} = u_1 = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n\int_{\beta_1}^{\beta_2} Ki_{n+2}(D\cos\beta)\cos^n\beta\, d\beta = \frac{2}{\Omega}\sum_{n=0}^{\infty}\frac{a_n}{D^n}\int_{x_2}^{x_1} Ki_{n+2}(x)\frac{x^n dx}{\sqrt{D^2 - x^2}},$$

$$m_1\xi_{12} = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n\int_{\beta_1}^{\beta_2} Ki_{n+1}(D\cos\beta)\cos^{n-1}\beta\, d\beta = \frac{2}{\Omega}\sum_{n=0}^{\infty}\frac{a_n}{D^{n-1}}\int_{x_2}^{x_1} Ki_{n+1}(x)\frac{x^{n-1}dx}{\sqrt{D^2 - x^2}}.$$

Case C. $\cos\Theta_2 = \cos\alpha\cos\beta$, $x = \tau/\cos\beta$, where τ is the optical thickness of the layer.

$$\xi_{12} = u_1 = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n \int_{\beta_1}^{\beta_2} Ki_{n+2}\left(\frac{\tau}{\cos\beta}\right)\cos^n\beta\, d\beta =$$

$$= \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n \tau^{n+1} \int_{x_1}^{x_2} Ki_{n+2}(x)\frac{dx}{x^{n+1}\sqrt{x^2-\tau^2}},$$

$$m_1\xi_{12} = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n \int_{\beta_1}^{\beta_2} Ki_{n+1}\left(\frac{\tau}{\cos\beta}\right)\cos^{n-1}\beta\, d\beta =$$

$$= \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n \tau^n \int_{x_1}^{x_2} Ki_{n+1}(x)\frac{dx}{x^n\sqrt{x^2-\tau^2}}.$$

In the limiting cases where $\beta_1 = -\pi/2$, $\beta_2 = \pi/2$ (for an infinite layer of medium)

$$\xi_{12} = u_1 = (2\pi/\Omega)\sum_{n=0}^{\infty} a_n E_{n+2}(\tau),$$

$$m_1\xi_{12} = (2\pi/\Omega)\sum_{n=0}^{\infty} a_n E_{n+1}(\tau). \qquad (10)$$

The properties of function $E_n(\tau)$ have been fully described in [7].

Case D. $\cos\Theta_2 = \cos\alpha\sin\beta$, $x = x_0/\sin\beta$, where x_0 is the optical distance to plane 2.

$$\xi_{12} = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n \int_{\beta_1}^{\beta_2} Ki_{n+2}\left(\frac{x_0}{\sin\beta}\right)\sin^{n-1}\beta\cos\beta\, d\beta =$$

$$= \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n x_0^n \int_{x_2}^{x_1} Ki_{n+2}(x)\frac{dx}{x^{n+1}},$$

$$m_1\xi_{12} = \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n \int_{\beta_1}^{\beta_2} Ki_{n+1}\left(\frac{x_0}{\sin\beta}\right)\sin^{n-1}\beta\, d\beta =$$

$$= \frac{2}{\Omega}\sum_{n=0}^{\infty} a_n x_0^n \int_{x_2}^{x_1} Ki_{n+1}(x)\frac{dx}{x^n\sqrt{x^2-x_0^2}},$$

$$u_1 = \frac{2x_0}{\Omega}\sum_{n=0}^{\infty} a_n \int_{x_2}^{x_1} Ki_{n+2}(x)\frac{(x^2-x_0^2)^{(n-1)/2}}{x^{n+1}}dx.$$

By successive application of formulas (4) and (5) we can obtain for ξ_{12} an algebraic series without integrals. According to (6), in the case of $\alpha_0 > 0$, when $x_0 \to 0$, $x_2 \to 0$, $x_1 \to \infty$, $\xi_{12} \to \infty$.

The problem posed with respect to angular coefficients for $f(\Theta) = \cos\Theta$ has been solved in detail in several papers by Mikk [8, 9]. The solution is ultimately obtained by means of tables of Bessel functions and, hence, is suitable for very simple engineering calculations. An analysis of the particular solution proposed here for $f(\Theta) = \cos\Theta$ gives different expressions for the main functions used by Mikk:

$$M(x) = (4/\pi)Ki_3(x)$$

instead of

$$M(x) = (2/\pi)\left[(1+x^2)\int_x^{\infty} K_0(u)\,du + xK_0(x) - x^2K_1(x)\right],$$

$$N_1(x) = (4/\pi)[Ki_1(x) - Ki_3(x)]$$

instead of

$$N_1(x) = (2/\pi)\left[(1-x^2)\int_x^{\infty} K_0(u)\,du - xK_0(x) + x^2K_1(x)\right],$$

$$N_2(x) = (4/3\pi)[Ki_2(x) - x(Ki_1(x) - Ki_3(x))]$$

instead of

$$N_2(x) = (2x/3\pi)\left[(2-x^2)K_1(x) + xK_0(x) - (3-x^2)\int_x^{\infty} K_0(u)\,du\right].$$

The expression for function $S_1(x)$ is the same:

$$S_1(x) = 2E_3(x)$$

[see (1) and (10)].

The expressions in terms of Bessel functions and integrals of them are a little more complex, the more the existing tables [10] are complicated by exponential functions. The tables of $Ki_n(x)$ functions, however, are much less thorough. On the other hand, reference to the tables allows only manual calculation (using a mechanical calculator), which is inapplicable in the case of analytically assigned complex configurations of the volume of the medium.

It follows from these remarks that the use of the $Ki_n(x)$ functions can be much more effective if they are approximated by simple and accurate approximate formulas. The series derived from function $K_0(x)$ and used in [5] converge slowly. Primak [4] suggested using a series for $Ki_1(x)$ obtained by expanding the integrand. The formulas for $Ki_2(x)$ and subsequent formulas are formed from $Ki(x)$ by integration according to (2):

$$Ki_1(x) = \pi/2 - 1.1159315x - 0.1265783x^3 +$$
$$+ (1 + x^2/12)\,0.9986x\ln x;$$

$$Ki_2(x) = 1 - \pi x/2 + 0.8079257x^2 +$$
$$+ 0.0364914x^4 - 0.4993(1 + x^2/24)x^2\ln x;$$

$$Ki_3(x) = \pi/4 - x + \pi x^2/4 - 0.3248642x^3 - 0.0080870x^5 +$$
$$+ 0.16643(1 + x^2/40)x^3\ln x;$$

$$Ki_4(x) = 2/3 - \pi x/4 + 0.5x^2 - \frac{\pi}{12}x^3 +$$
$$+ 0.0916327x^4 + 0.0014591x^6 -$$
$$- 0.0416(1 + x^2/60)x^4\ln x.$$

The coefficients of x^k with the highest power are corrected so that exact values of the functions Ki_n (1) are obtained. The factor 0.9986 in the last term of the first formula is chosen to reduce the error. The error in the range $0 \le x < 1.2$ is of tenths of one per cent and decreases with increase in the order of the function. In practice the values of the optical thicknesses usually lie in this range if the radiation incident at large angles (from distant regions) is ignored. For large x Primak [4] proposes formulas with exponential functions.

The reduction of three-dimensional problems to two-dimensional problems required investigation of the incomplete integrals $Ki_n(x, \alpha_0)$, where α_0 is the limiting angle. The larger x and n and the closer the value of α_0 to $\pi/2$, the smaller will be the error due to inclusion of radiation from nonexistent regions in the interval $(\alpha_0, \pi/2)$.

In conclusion we must once again point out that all the formulas here are for direct fluxes. Reradiation and scattering of the fluxes can be taken into account by solution of the integral or equivalent equations. The calculation of the coefficients presented here is an essential preliminary step.

NOTATION

Θ—angle between normal to element of surface and ray; dF_1 and dF_2—surface elements; l, m—distance between them; β—angle between normal to dF and projection of ray on cross section of two-dimensional body; α—angle between this projection and the ray, $\cos\Theta = \cos\alpha\cos\beta$; $f(\Theta)$—emission characteristic; a_n—dimensionless coefficients; Ω—equivalent solid angle determined from (1); k, m^{-1}—coefficient of attenuation of ray; x—optical distance with respect to attenuation of ray; $d\omega$—solid angle at which element dF_2 is seen from point dF_1; $Ki_n(x) = \int_0^{\pi/2} \exp(-x/\cos\alpha)\cos^{n-1}\alpha\, d\alpha$; $E_n(x) = \int_1^{\infty} \exp(-xt) \times t^{-n}\, dt$; ξ_{12}—coefficient of irradiation for point dF; φ_{12}—mean coefficient of irradiation; L—extent of surface F_1; q_{eff2}, W/m^2—density of hemispherical effective emission of surface 2; η_{i1}, W/m^2—density of bulk incident flux at point dF_1; m—coefficient of spatial distribution of incident direct flux; u—coefficient of utilization of source; α_0—limiting angle for three-dimensional system.

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